$\Gamma\backslash G$, $\quad G = SL_2(\mathbb{R})$ $\qquad m \in \mathbb{Z}$

$$f_s\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}y & \\ & y^{-1}\end{pmatrix} k_\theta\right) = |y|^{1+s}\, e(m\theta)$$ holom. for $\mathrm{Re}(s)>1$

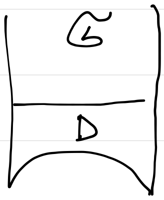
$$\rightsquigarrow \quad E_s : \Gamma\backslash G \longrightarrow \mathbb{C} \quad \text{for} \quad \mathrm{Re}(s) > 1$$
$$g \longmapsto \sum_{\Gamma_P\backslash\Gamma} f_s(\gamma g)$$

$$E_{s,P} = f_s + c(s) f_{-s}$$

Theorem $\quad E_s$, $c(s)$: merom. cont. to $\mathbb{C}$,
$$E_{-s} = c(-s)\, E_s, \qquad c(-s)\,c(s) = 1.$$

First step today: explain how to recover E_s from $c(s)$.
(Then, meromorphic continuation of $c(s)$ $\Longrightarrow$ that of E_s.)

Notation recap



$$H_D := \left\{ \varphi \in L^2 := L^2(\Gamma\backslash G) : \ K\text{-type } m, \ \varphi_P|_{\mathfrak{S}} = 0 \right\}$$ closed $\subseteq L^2$

$$\Lambda\varphi := \varphi - \chi\varphi_P, \qquad \chi = 1_{\mathfrak{S}}$$

$$\rightsquigarrow \quad \Lambda : L^2 \to H_D \quad \text{orth. projection}$$

Defn For $U \underset{\text{open}}{\subseteq} \mathbb{C}$, $\mathcal{M}(U) := \{\text{meromorphic } U \to \mathbb{C}\}$

and for V: nice topological vector space, $\mathcal{M}(U \to V) := (\ldots)$.

analytic after multiplying by some nonzero polynomial, locally in U
(see beginning of §11 of Borel)

Let $\mu = (\mu_\pm)_\pm$, $\mu_\pm \in \mathcal{M}(U)$.

Define, for $s \in U$, $\Psi_{\mu,s} : \Gamma \backslash G \to \mathbb{C}$

$$x \mapsto \underbrace{1_{\tilde{G}}(x)}_{\chi(x)} \sum_\pm \mu_\pm(s) f_{\pm s}(x)$$

i.e., $\Psi_{\mu,s} = \chi \sum_\pm \mu_\pm(s) f_{\pm s}.$

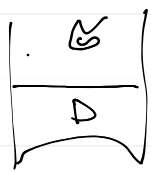
<u>Example</u> If $\mu = \underset{\text{as in } E_{s,\phi}}{(1, c)}$ $\left(\begin{array}{l}\mu_+ = 1\\ \mu_- = c\end{array}\right)$

Then $\Psi_{\mu,s} = \chi(E_{s,\phi})$ $= \chi(f_s + c(s) f_{-s})$, and so

$$e_s := \Lambda(E_s) = E_s - \Psi_{\mu,s} \in H_D$$

$$E_s = e_s + \Psi_{\mu,s}$$

Thus: $\Psi_{\mu,s}$: supported here, may grow at ∞



e_s: "concentrated" here, decays rapidly near ∞ (in $\tilde{G}$)

<u>Observation</u> For $\alpha \in I_c^\infty(G) = \{\alpha : \alpha(kxk^{-1})\ \forall k \in K\} \subseteq C_c^\infty(G)$

$$\Lambda(E_s * \alpha) = \Lambda(\hat{\alpha}(s) E_s) = \hat{\alpha}(s)\, \Lambda E_s = \hat{\alpha}(s)\, e_s$$

$$\| $$

$$\Lambda(e_s * \alpha) + \underbrace{\Lambda(\Psi_{\mu,s} * \alpha)}_{\in H_D \text{ (final lemma of Lecture 10)}}$$

$$\Rightarrow (\hat{\alpha}(s) - \Lambda \circ (*\alpha)) e_s = \Lambda(\Psi_{\mu,s} * \alpha) \in H_D$$

Idea: invert this to determine e_s, hence E_s, in terms of $c(s)$.

Fredholm theory $\Rightarrow$ $[U \ni s \mapsto (\hat{\alpha}(s) - \Lambda \circ (*\alpha))^{-1}]$
$U \subseteq \{s : \mathrm{Re}(s) > 1\}$ $\in \mathcal{M}(U \to \mathcal{L}(H_D))$.

Fredholm theory Let H: Hilbert space.
$(,)$
$\mathcal{L}(H) := \{$ bounded linear ops $H \to H \}$
: Banach space, $\|\cdot\|$ = operator norm

Let $T : H \to H$ be a densely defined linear operator.

Resolvent set $\rho(T) := \left\{ \lambda \in \mathbb{C} \text{ s.t. } \lambda - T : H \to H \text{ has dense image, bounded inverse} \right\}$

$=: R(\lambda, T)$
$=: (\lambda - T)^{-1}$

Spectrum $sp(T) := \mathbb{C} - \rho(T)$
$\cup\!\!\mid$

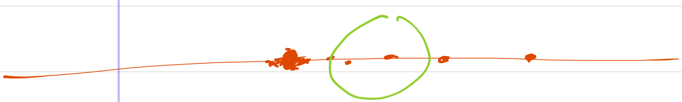
Discrete spectrum $\sigma(T) := \{ \lambda \in \mathbb{C} : \ker(\lambda - T) \neq 0 \}$
$= \{$ eigenvalues for T in $H \}$

If T: closed, then $\rho(T)$: open, $sp(T)$: closed,

and
$$\begin{aligned} \rho(T) &\longrightarrow \mathcal{L}(H) \\ \lambda &\longmapsto (\lambda - T)^{-1} \end{aligned}$$
is holomorphic.

If T: compact, then $sp(T) \subseteq \{0\} \cup \sigma(T)$

spectral theory of compact operators
+ Taylor series



$$(\lambda - T)^{-1} = \lambda^{-1}(1 - T/\lambda)^{-1}$$
$$= \lambda^{-1} \sum_{j \geq 0} \lambda^{-j} T^{j}.$$

If moreover T: self-adjoint, then $sp(T) \subseteq \mathbb{R}$
$\Rightarrow [\lambda \mapsto (\lambda - T)^{-1}]$: holom. on $\mathbb{C} - \mathbb{R}$.

Lemma Let $\underset{\text{open}}{U} \subseteq \mathbb{C}$, $\mu_\pm \in \mathcal{M}(U)$.

Let $\alpha \in {}_0^* I_c^\infty(G)$ s.t. $|\hat{\alpha}(s)| \geq 1/2$ $\forall s \in U$. $\rightsquigarrow$ $\Phi_{\mu,s} = \chi \sum \mu_\pm(s) f_{\pm s}$. $(s \in U)$

$\exists$ meromorphic $[U \ni s \mapsto F_{\mu,s}] : U \to \left\{ \begin{array}{l} \text{moderate growth functions} \\ \text{on } \Gamma\backslash G, \text{ K-type } m \end{array} \right\}$

s.t. (i) $F_{\mu,s} - \Phi_{\mu,s} =: g_{\mu,s} \in H_D$, $[s \mapsto g_{\mu,s}] \in \mathcal{M}(U \to H_D)$.

(ii) $\Lambda(F_{\mu,s} * \alpha) = \hat{\alpha}(s)\, \Lambda(F_{\mu,s})$

(**Ex** $\mu = (1, c)$, $U \subseteq \{\mathrm{Re}(s) > 1\}$, then $F_{\mu,s} = E_s$.)

Proof Uniqueness Assume $F_{\mu,s}$ satisfies (i) + (ii).

$$\hat{\alpha}(s) \underbrace{\Lambda(F_{\mu,s})}_{\Lambda(g_{\mu,s})} \overset{(ii)}{=} \Lambda(F_{\mu,s} * \alpha) \overset{(i)}{=} \Lambda(\Phi_{\mu,s} * \alpha) + \Lambda(g_{\mu,s} * \alpha)$$

$$\Rightarrow \left(\hat{\alpha}(s) - \Lambda \circ (*\alpha)\right) g_{\mu,s} = \Lambda(\Phi_{\mu,s} * \alpha) \in H_D$$

$\underset{\text{Fredholm theory}}{\Longrightarrow}$ $g_{\mu,s}$ is determined by $\Phi_{\mu,s}$, α.

$\Longrightarrow$ $F_{\mu,s}$ —— // ——.

Existence Fredholm theory.

Prop Let $U \underset{\text{open}}{=} \{\mathrm{Re}(s) > 1\}$, $\alpha \in I_c^\infty(G)$ s.t. $|\hat{\alpha}| \geq 1/2$ on U.

Then $(1, c)$ (as in $E_{s,P}$) is the unique solution $\mu = (\mu_\pm)_\pm$, $\mu_\pm \in \mathcal{M}(U)$ to the system of linear equations (over $\mathcal{M}(U)$)

(1) $\mu_+ = 1$

(2) The function $F_{\mu,s}$, constructed in the Lemma, satisfies
$$\left(\mathcal{C} - \frac{s^2-1}{2}\right) F_{\mu,s} = 0.$$
distributionally, i.e., when tested against C_c^∞

For this μ, we have $F_{\mu,s} = E_s$.

Proof Note: if $\mu = (1,c)$, $F_{\mu,s} = E_s$ satisfies the above.

Suppose μ has the above properties.

$$\rightarrow F_{\mu,s}: \left.\begin{array}{l} K\text{-type } m, \\ \text{moderate growth}, \\ \mathcal{C}\text{-eigenvalue } \frac{s^2-1}{2} \end{array}\right\} \Rightarrow F_{\mu,s} : \text{aut. form}$$

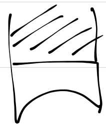
(Want: $\mu = (1,c)$, $F_{\mu,s} = E_s$.)

Set $R_s := E_s - F_{\mu,s}$: aut. form

decreases as height $\to \infty$ for $\mathrm{Re}(s) > -1$ $\Leftarrow \mathrm{Re}(s) > 1$

$$R_{s,P} = \underbrace{E_{s,P}}_{f_s + c(s) f_{-s}} - \underbrace{F_{\mu,s,P}}_{\overset{\varphi}{=}\ \underbrace{\mu_+(s)}_{=1} f_s + \mu_-(s) f_{-s}} = (c(s) - \mu_-(s)) f_{-s}$$

at height $\geq T$



$R_s = 0 \ \forall s.$

$\Uparrow$

$R_s = 0 \ \forall s \notin \mathbb{R}$, $\mathrm{Re}(s) > 1$

$\Uparrow$

$\Rightarrow R_s$: bounded on $\Gamma \backslash G$,

hence in L^2. $\Longrightarrow \frac{s^2-1}{2} \in \mathbb{R} \iff s \in \mathbb{R} \cup i\mathbb{R} \underset{\mathrm{Re}(s)>1}{\Longrightarrow} s \in \mathbb{R}$ OR $R_s = 0$

Sublemma, $\mathcal{C} R_s = \frac{s^2-1}{2} R_s$

Sublemma Let $\varphi \in C^\infty(\Gamma \backslash G)$ s.t. each derivative is in L^2.

Suppose $\mathcal{C}\varphi = \lambda \varphi$. Then $\lambda \in \mathbb{R}$ OR $\varphi = 0$

Sublemma Let $\varphi \in C^\infty(\Gamma \backslash G)$ s.t. each derivative is in L^2.
Suppose $\mathcal{C}\varphi = \lambda\varphi$. Then $\lambda \in \mathbb{R}$ OR $\varphi = 0$

Proof

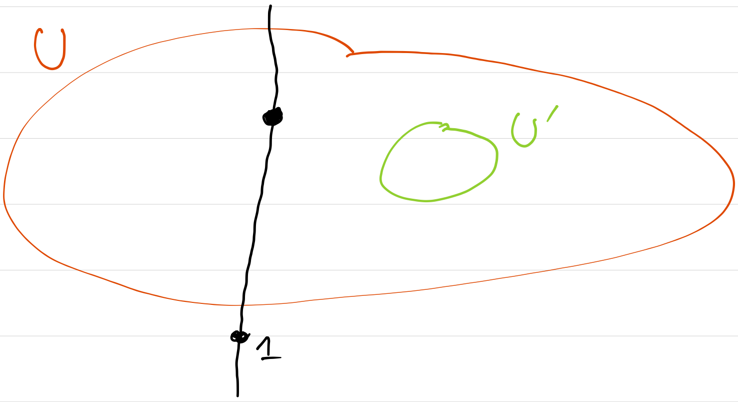
$$\langle g\varphi_1, g\varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle \quad \forall g \in G$$

$$\Longrightarrow \quad \langle X\varphi_1, \varphi_2 \rangle = -\langle \varphi_1, X\varphi_2 \rangle \quad \forall X \in \mathfrak{g}$$

$$\mathcal{C} = \tfrac{1}{2}H^2 + EF + FE.$$

$$\Rightarrow \quad \lambda\langle \varphi, \varphi \rangle = \langle \mathcal{C}\varphi, \varphi \rangle = \langle \varphi, \mathcal{C}\varphi \rangle = \bar{\lambda}\langle \varphi, \varphi \rangle. \square$$

Meromorphic continuation (of $c(s)$, hence of E_s)



$|\hat{\alpha}| \geq 1/2$ on U

(see Borel, §11.4?)

Consider the system of linear equation over $\mathcal{M}(U)$ given in the Prop.

We've seen that this system, "restricted to U'", has as its unique solution $(1, c)$.

$$Ax = b, \quad \text{solve for } x$$

Find a nonzero minor of A of same size as b.
Reduce to case $A = \square$ matrix.
Then $x = A^{-1}b$, A^{-1} = computed via Cramer's rule.